A CLASS-BOOK OF NON-EUCLIDEAN GEOMETRY.

By DR. GEORGE BRUCE HALSTED.

America has taken a step in advance of all the world. She now has a text-book, a manual for class use, in non-Euclidean geometry. It is a pleasure to point out the close connection of THE AMERICAN MATHEMATICAL MONTHLY with this note-worthy achievement.

In this book, *Non-Euclidean Geometry*, by Henry Parker Manning, Ph. D., Assistant Professor of Pure Mathematics in Brown University, Ginn & Co., 1901, Pp. v+95, the method of treatment that has been taken as the basis of the first chapter, and which consequently underlies the other three chapters of the book, is that of Saccheri, drawn directly and solely from the pages of THE AMERICAN MATHEMATICAL MONTHLY where the first translation of Saccheri began to be published in June, 1894.

THE MONTHLY, Vol. I, p. 188, gives Saccheri's first proposition literally: "If two equal straights [sects] AC, BD, make with the straight AB angles equal toward the same parts: I say the angles at the join CD will be mutually equal." On page 189 is "Proposition II. The quadrilateral $ABCD$ remaining the same, the sides AB, CD are bisected in points M, and H. I say the angles at the join MH will be on both sides right."

Professor Manning paraphrases these two together on page 5: "If two equal lines in a plane are erected perpendicular to a given line, the line joining their extremities makes equal angles with them and is bisected at right angles by a third perpendicular erected midway between them."

In some respects we prefer the phraseology of Saccheri to that of Manning. Like the French, neither possesses a word for a piece of a straight line, German strecke, English *sect*. There is a kind of unbounded entity such that one and only one is on two points. This may be called *the straight*. It appears as an element in projective geometry. But it is of the essence of metric geometry that two points shall also completely determine an entity bounded by them, *the sect*, with which the idea of precise individual magnitude or quantity may be connected by setting up a conventional system of measurement. Distance is the result of the comparison of two sects. The distance between two points is the length of their sect in terms of a standard sect, say the centimeter.

Both the accepted popular and the accepted mathematical definitions of distance make it always a number, as, *e. g.* Wentworth, 1899, page 8, §50. "Def. The *distance* between two points is *the length* of the straight line [sect] that joins them;" and again the Cayley-Klein definition: "The distance between two points is equal to a constant times the logarithm of the cross-ratio in which the line joining the two points is divided by the fundamental quadric." Saccheri calls the two equal sects of his first proposition *straights*. Manning goes farther off, calling them *lines*.

B. A. W. Russell in "An Essay on the Foundations of Geometry" uses

the word distance' as a confounding and confusing designation for a sect itself and also for the numerical measures of that sect, whether by superposition, ordinary ratio, indeterminate as depending on the choice of a unit, or by projective metrics, indeterminate as depending on the fixing of the three points to be taken as constant in the varying cross ratios, these cross ratios themselves to be defined as numbers by the method of von Staudt, without presupposing ordinary measurement.

The confusion which may thus be introdued just from lack of a word is powerfully shown by the illustrious Poincaré in a response to Mr. Russell in the *Revue de Métaphysique et de Morale*, 1900, pp. 73—86, entitled "Sur les principes de la Géométrie." The following four sentences are curious in showing his results and at the same time showing the lack in French which causes a borrowing from German: "Comme le fait remarquer M. Halsted dans une brochure récente (*Science*, N. S. Vol. X, No. 251), M. Russell a eu tort d'employer indifférement le mot distance pour désigner ce que les Allemands appelent 'Strecke,' et en même temps la mesure de cette 'Strecke.' Le nom de distance ne convient qu' à la mesure de cette 'Strecke,' et cette mesure ne peut être définie que par une convention.

Si M. Russell n' avait pas, comme le lui reproche M. Halsted (vide supra), employé le mot distance dans deux sens différents, il n' y aurait plus d' apparence de cercle vicieux. Où serait cette apparence si l' on avait écrit: La distance est le résultat de la comparison de deux 'Strecken'."

Again, in both propositions Saccheri speaks of the *join* of two points. Manning paraphrases it as "the line joining" the two points. In a note to Euclid I. 5, Todhunter says of the phrase "Join FC"; "Custom seems to allow this singular expression as an abbreviation for 'draw the straight line FC', or for 'join F to C by the straight line FC.'" In Saccheri the join AB means the sect terminated by A and B. In projective geometry the join AB means the unbounded straight on A and B.

Under the heading *Definitions*, Saccheri says: "Since (from our first) the straight joining the extremities of equal perpendiculars standing upon the same straight (which we will call base), makes equal angles with these perpendiculars, three hypotheses are to be distinguished according to the species of these angles. And the first, indeed, I will call hypothesis of right angle; the second, however, and the third I will call hypothesis of obtuse angle, and hypothesis of acute angle." This Manning paraphrases as follows, under the heading *The Three Hypotheses*: "The angles at the extremities of two equal perpendiculars are either right angles, acute angles, or obtuse angles, at least for restricted figures. We shall distinguish the three cases by speaking of them as the hypothesis of the right angle, the hypothesis of the acute angle, and the hypothesis of the obtuse angle, respectively."

Saccheri's Prop. III. is: "If two equal straights, AC, BD, stand perpendicular to any straight, AB: I say the join CD will be equal, or less, or greater than AB, according as the angles at CD are right, or obtuse, or acute."

This Manning paraphrases as follows: "The line joining the extremities of two equal perpendiculars is, at least for any restricted portion of the plane, equal to, greater than, or less than the line joining their feet in the three hypotheses respectively."

In the same way is paraphrased Saccheri's Prop. IV., the converse of III.

Saccheri's Corollary about quadrilaterals with three right angles is given, page 12.

Saccheri's Prop. V. is: "The hypothesis of right angle, if even in a single case it is true, always in every case alone is true." In giving this, Manning writes: "If the hypothesis of *a* right angle," &c, evidently a slip for *the* right angle. Of course the Latin has no article.

Prop. VI. and Prop. VII. are combined, p. 13. Prop. IX. is reproduced on p 14. Prop. X. is given on p. 9.

In Prop. XI. Saccheri with the hypothesis of right angle demonstrates the celebrated Postulatum of Euclid, thus showing that his hypothesis of right angle is the ordinary Euclidean geometry. Manning does not reproduce this demonstration but says, p. 27: "The three hypotheses give rise to three systems of Geometry, which are called the Parabolic, the Hyperbolic, and the Elliptic Geometries. They are also called the Geometries of Euclid, of Lobachevski, and of Riemann." It should be noted that Manning's book gives only the simple elliptic, or single elliptic, or Clifford-Klein Geometry. It never even mentions the double elliptic or Spherical or Riemannian Geometry, which Killing maintains was the only form which ever came before Riemann's mind.

Manning's Chapter II., The Hyperbolic Geometry, seems taken bodily from Halsted's translation of Lobachevski's "Geometrical Researches on the Theory of Parallels." Though Halsted's translation of Bolyai is specifically mentioned on page 94, yet Manning shows no signs of having read it, and thus his book is confined within the bounds of propædeutics.

The most extraordinary two dozen pages in the history of thought is "The Science Absolute of Space," by Bolgai János. This is the most perfect case of genius. Take as example his §34: Through a given point to draw a parallel to a given straight. So simple; and yet neither Lobachevski nor anyone else ever reached it. It seems supernatural, uncanny. it makes one's hair stand on end. Perhaps he was, as he called himself, the phœnix of Euclid.

Manning's Chapter III., The Elliptic Geometry, pp. 62-8, is very brief, but what there is of it is good. The final Chapter IV., Analytic Non-Euclidean Geometry, is devoted to putting into coördinate and equational notation the new matters reached synthetically in the preceding chapters. The book ends with a Historical Note, pp. 91-5. This, in the main sound, may be in parts misleading. Thus it says, p. 91; "Legendre proved that the sum of the angles of a triangle can never exceed two right angles, and that if there is a single triangle in which this sum is equal to two right angles, the same is true of all triangles. This was, of course, on the supposition that a line is of infinite length." Now this beautiful theorem, if a single, then all triangles, Manning has in his own

book, p. 12, under the form "If the hypothesis of [a] right angle is true in a single case, it holds true in every case." proved by Saccheri without any supposition on the length of the line and a century before Legendre.

Again, not even the name of Schweikart is mentioned, though as I have shown in THE AMERICAN MATHEMATICAL MONTHLY, Vol. VII., pp. 247-252, and in "Science," Vol. XII., pp. 842-846, Schweikart may be considered the first to publish a genuine conscious treatise on Non-Euclidean Geometry [which I there give for the first time in English]. This fixes the date of the first conscious creation and naming of the Non-Euclidean Geometry as between 1812 and 1816.

We may perhaps timidly hope that the great Jesuit, Saccheri, had some suspicion of what he had really done. But meanwhile it seems almost certain that he really believed that his beautiful Non-Euclidean Geometry was all a reductio-ad-absurdum, and that he had really justified the title of his book, "Euclid Vindicated from Every Fleck," by proving Non-Euclidean Geometry untenable. On the other hand we of the new school, followers of Schweikart and Bolgai János, believe that it is our Non-Euclidean Geometry itself which finally vindicates Euclid from every fleck, and justifies the weighty tribute of Professor Alfred Baker: "Of the perfection of Euclid (B. C. 290) as a scientific treatise, of the marvel that such a work could have been produced two thousand years ago, I shall not here delay to speak. I content myself with making the claim that, as a historical study, Euclid is, perhaps, the most valuable of those that are taken up in our educational institutions."

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DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

115. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah Iowa.

Find the conditions of the coefficients of a general biquadratic equation so that it may be solved by quadratics.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and the PROPOSER.

To solve as a pure quadratic, without using the cubic, we might proceed as follows:

Let $x^4+ax^3+bx^2+cx+d=0$ be the general biquadratic.

(1) Then $(x^2+\frac{1}{2}ax)^2+bx^2-\frac{1}{4}a^2x^2+cx+d=0$.

Now if $c=\frac{1}{2}ab-\frac{1}{8}a^3$, we get $(x^2+\frac{1}{2}ax)^2+(b-\frac{1}{4}a^2)(x^2+\frac{1}{2}ax)+d=0$.